On feasibility, stability and performance in distributed model predictive control

Pontus Giselsson and **Anders Rantzer**
Department of Automatic Control
Lund University
Box 118, SE-221 00 Lund, Sweden
{pontusg,rantzer}@control.lth.se

Abstract—In distributed model predictive control (DMPC), where a centralized optimization problem is solved in distributed fashion using dual decomposition, it is important to keep the number of iterations in the solution algorithm, i.e. the amount of communication between subsystems, as small as possible. At the same time, the number of iterations must be enough to give a feasible solution to the optimization problem and to guarantee stability of the closed loop system. In this paper, a stopping condition to the distributed optimization algorithm that guarantees these properties, is presented. The stopping condition is based on two theoretical contributions. First, since the optimization problem is solved using dual decomposition, standard techniques to prove stability in model predictive control (MPC), i.e. with a terminal cost and a terminal constraint set that involve all state variables, do not apply. For the case without a terminal cost or a terminal constraint set, we present a new method to quantify the control horizon needed to ensure stability and a prespecified performance. Second, the stopping condition is based on a novel adaptive constraint tightening approach. Using this adaptive constraint tightening approach, we guarantee that a primal feasible solution to the optimization problem is found and that closed loop stability and performance is obtained. Numerical examples show that the number of iterations needed to guarantee feasibility of the optimization problem, stability and a prespecified performance of the closed-loop system can be reduced significantly using the proposed stopping condition.

Index Terms—Distributed model predictive control, performance guarantee, stability, feasibility

I. INTRODUCTION

Distributed model predictive control (DMPC) can be divided into two main categories. In the first category, local optimization problems that are solved sequentially and that take neighboring interaction and solutions into account, are solved in each subsystem. This is done in [1] for linear systems and in [2] for nonlinear systems. In [3] a DMPC scheme is presented in which stability is proven by adding a constraint to the optimization problem that requires a reduction of an explicit control Lyapunov function. In [4], [5] stability is guaranteed for systems satisfying a certain matching condition and if the coupling interaction is small enough. In the second category, to which the current paper belong, a centralized optimization problem with a sparse structure is solved using a distributed optimization algorithm. This approach is taken in [6] where stability is guaranteed in every algorithm iteration. A drawback to this method is that full model knowledge is assumed in each node. Other approaches in the DMPC literature rely on dual decomposition to solve the centralized MPC problem in

distributed fashion. This approach is taken in, e.g. [7], [8], [9], where a (sub)gradient algorithm is used to solve the dual problem and in [10] where the algorithm is based on the smoothing technique presented in [11]. Among these, the only stability proof is given in [12], [9], where a terminal point constraint is set to the origin, which is very restrictive.

One reason for the lack of stability results in DMPC based on dual decomposition, is that the standard techniques to prove stability in MPC do not apply. In MPC, terminal costs and terminal constraint sets that involve all state variables are used to show stability of the closed loop system, see [13], [14]. This is not compatible with dual decomposition. However, results for stability in MPC without a terminal constraint set or a terminal set, which fits also the DMPC framework used here, are available [15], [16]. In [16], a method to quantify the minimal control horizon that guarantees stability and a prespecified performance is presented. This is based on relaxed dynamic programming [17], [18] and a controllability assumption on the stage costs. In the current paper, we take a similar approach to quantify the control horizon needed to guarantee stability and a prespecified performance. The advantages of our approach over the one in [16] are twofold; we can, by solving a mixed integer linear program (MILP), verify our controllability assumption, further we get an explicit expression that relates the parameter in the controllability assumption with the obtained closed loop performance.

Besides the stability result, the main contribution of this paper is a stopping condition for DMPC controllers that use a distributed optimization algorithm based on dual decomposition. We use the distributed algorithm presented in [19], but any duality-based distributed algorithm, such as the standard dual ascent or ADMM [20], can be used. These duality based algorithms suffer from that primal feasibility is only guaranteed in the limit of iterations. Constraint tightening, which was originally proposed for robust MPC in [21], can also be used to generate feasible solutions within finite number of iterations, see [22]. However, the introduction of constraint tightening complicates stability analysis since the optimal value function without constraint tightening is used to show stability, while the optimization is performed with tightened constraints. This problem is addressed in [22] by assuming that the difference between the optimal value functions with and without constraint tightening is bounded by a constant. However, to actually compute such a constant is very difficult. The stopping condition in this paper is based on a novel adap-

tive constraint tightening approach that ensures feasibility w.r.t. the original constraint set with a finite number of algorithm iterations. In addition, the amount of constraint tightening is adapted until the difference between the optimal value functions with and without constraint tightening is bounded by a certain amount. This adaptation makes it possible to guarantee, besides feasibility of the optimization problem, also stability of the closed-loop system, without stating additional, unquantifiable assumptions.

The paper is organized as follows. In Section II we introduce the problem and present the distributed optimization algorithm in [19]. In Section III the stopping condition is presented and feasibility, stability, and performance is analyzed. Section IV is devoted to computation of a controllability parameter in the controllability assumption. A numerical example that shows the efficiency of the proposed stopping condition, is presented in Section V. Finally, in Section VI we conclude the paper.

II. PROBLEM SETUP AND PRELIMINARIES

We consider linear dynamical systems of the form

$$x_{t+1} = Ax_t + Bu_t, \qquad x_0 = \bar{x} \qquad (1)$$

where $x_t \in \mathbb{R}^n$ and $u_t \in \mathbb{R}^m$ denote the state and control vectors at time t and the pair (A, B) is assumed controllable. We introduce the following state and control variable partitions

$$x_t = [(x_t^1)^T, (x_t^2)^T, \dots, (x_t^M)^T]^T, \qquad (2)$$
$$u_t = [(u_t^1)^T, (u_t^2)^T, \dots, (u_t^M)^T]^T \qquad (3)$$

where the local variables $x_t^i \in \mathbb{R}^{n_i}$ and $u_t^i \in \mathbb{R}^{m_i}$. The A and B matrices are partitioned accordingly

$$A = \begin{pmatrix} A_{11} & \cdots & A_{1M} \\ \vdots & \ddots & \vdots \\ A_{M1} & \cdots & A_{MM} \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & \cdots & B_{1M} \\ \vdots & \ddots & \vdots \\ B_{M1} & \cdots & B_{MM} \end{pmatrix}.$$

These matrices are assumed to have a sparse structure, i.e., some $A_{ij} = 0$ and $B_{ij} = 0$ and the neighboring interaction is defined by the following sets

$$\mathcal{N}_i = \{j \in \{1, \dots, M\} \mid \text{if } A_{ij} \neq 0 \text{ or } B_{ij} \neq 0\}$$

for $i = 1, \dots, M$. This gives the following local dynamics

$$x_{t+1}^i = \sum_{j \in \mathcal{N}_i} \left(A_{ij} x_t^j + B_{ij} u_t^j \right), \qquad x_0^i = \bar{x}_i$$

for $i = 1, \dots, M$. The local control and state variables are constrained, i.e., $u^i \in \mathcal{U}_i$ and $x^i \in \mathcal{X}_i$. The constraint sets, $\mathcal{X}_i$, $\mathcal{U}_i$ are assumed to be bounded polytopes containing zero in their respective interiors and can hence be represented as

$$\mathcal{X}_i = \{x^i \in \mathbb{R}^{n_i} \mid C_x^i x^i \leq d_x^i\},$$
$$\mathcal{U}_i = \{u^i \in \mathbb{R}^{m_i} \mid C_u^i u^i \leq d_u^i\}$$

where $C_x^i \in \mathbb{R}^{n_{c_{x^i}} \times n_i}$, $C_u^i \in \mathbb{R}^{n_{c_{u^i}} \times m_i}$, $d_x^i \in \mathbb{R}_{>0}^{n_{c_{x^i}}}$ and $d_u^i \in \mathbb{R}_{>0}^{n_{c_{u^i}}}$. We also denote the total number of linear inequalities describing all constraint sets by $n_c := \sum_{i=1}^M \left(n_{c_{x^i}} + n_{c_{u^i}} \right)$. The global constraint sets are defined from the local ones through

$$\mathcal{X} = \mathcal{X}_1 \times \dots \times \mathcal{X}_M, \qquad \mathcal{U} = \mathcal{U}_1 \times \dots \times \mathcal{U}_M.$$

We use a separable quadratic stage cost

$$\ell(x, u) = \sum_{i=1}^M \ell_i(x^i, u^i) = \frac{1}{2} \left(\sum_{i=1}^M (x^i)^T Q_i x^i + (u^i)^T R_i u^i \right)$$

where $Q_i \in \mathbb{S}_{++}^{n_i}$ and $R_i \in \mathbb{S}_{++}^{m_i}$ for $i = 1, \dots, M$ and $\mathbb{S}_{++}^n$ denotes the set of symmetric positive definite matrices in $\mathbb{R}^{n \times n}$. The optimal infinite horizon cost from initial state $\bar{x} \in \mathcal{X}$ is defined by

$$\begin{aligned} V_\infty(\bar{x}) := \quad & \min_{x,u} \quad \sum_{t=0}^\infty \ell(x_t, u_t) \qquad (4) \\ & \text{s.t.} \quad x_t \in \mathcal{X} \quad, \quad u_t \in \mathcal{U} \\ & \qquad x_{t+1} = Ax_t + Bu_t \\ & \qquad x_0 = \bar{x}. \end{aligned}$$

Such infinite horizon optimization problems are in general intractable to solve exactly. A common approach is to solve the problem approximately in receding horizon fashion. To this end, we introduce the predicted state and control sequences $\{z_\tau\}_{\tau=0}^{N-1}$ and $\{v_\tau\}_{\tau=0}^{N-1}$ and the corresponding stacked vectors

$$\mathbf{z} = [z_0^T, \dots, z_{N-1}^T]^T, \qquad \mathbf{v} = [v_0^T, \dots, v_{N-1}^T]^T$$

where z_τ and v_τ are predicted states and controls τ time steps ahead. The predicted state and control variables z_τ, v_τ are partitioned into local variables as in (2) and (3) respectively. We also introduce the following stacked local vectors

$$\mathbf{z}_i = [(z_0^i)^T, \dots, (z_{N-1}^i)^T]^T, \quad \mathbf{v}_i = [(v_0^i)^T, \dots, (v_{N-1}^i)^T]^T.$$

Further, we introduce the tightened state and control constraint sets

$$(1-\delta)\mathcal{X}_i = \{x^i \in \mathbb{R}^{n_i} \mid C_x^i x^i \leq (1-\delta)d_x^i\},$$
$$(1-\delta)\mathcal{U}_i = \{u^i \in \mathbb{R}^{m_i} \mid C_u^i u^i \leq (1-\delta)d_u^i\}$$

where $\delta \in (0, 1)$ decides the amount of relative constraint tightening. The following optimization problem, which has neither a terminal cost nor a terminal constraint set, is solved in the DMPC controller for the current state $\bar{x} \in \mathbb{R}^n$

$$\begin{aligned} V_N^\delta(\bar{x}) := \quad & \min_{\mathbf{z}_t, \mathbf{v}_t} \quad \sum_{\tau=0}^{N-1} \ell(z_\tau, v_\tau) \\ & \text{s.t.} \quad z_\tau \in (1-\delta)\mathcal{X}, \, \tau = 0, \dots, N-1 \\ & \qquad v_\tau \in (1-\delta)\mathcal{U}, \, \tau = 0, \dots, N-1 \\ & \qquad z_{\tau+1} = Az_\tau + Bv_\tau, \, \tau = 0, \dots, N-2 \\ & \qquad z_0 = \bar{x}. \end{aligned}$$
$$(5)$$

By stacking all decision variables into one vector

$$\mathbf{y} = [z_0^T, \dots, z_{N-1}^T, v_0^T, \dots, v_{N-1}^T]^T \in \mathbb{R}^{(n+m)N} \quad (6)$$

the optimization problem (5) can more compactly be written as

$$\begin{aligned} V_N^\delta(\bar{x}) := \quad & \min_{\mathbf{y}} \quad \tfrac{1}{2}\mathbf{y}^T \mathbf{H} \mathbf{y} \qquad (7) \\ & \text{s.t.} \quad \mathbf{A}\mathbf{y} = \mathbf{b}\bar{x} \\ & \qquad \mathbf{C}\mathbf{y} \leq (1-\delta)\mathbf{d} \end{aligned}$$

where $\mathbf{H} \in \mathbb{S}_{++}^{(n+m)N}$, $\mathbf{A} \in \mathbb{R}^{n(N-1) \times (n+m)N}$, $\mathbf{b} \in \mathbb{R}^{n(N-1) \times n}$, $\mathbf{C} \in \mathbb{R}^{n_c N \times (n+m)N}$ and $\mathbf{d} \in \mathbb{R}_{>0}^{Nn_c}$ are built accordingly. Such sparse optimization problems can be solved

in distributed fashion using, e.g., the classical dual ascent, the alternating direction of multipliers method (ADMM) [20], or the recently developed algorithm in [19]. The algorithm in [19] is a dual accelerated gradient algorithm and is used in the current paper for simplicity. Distribution of these methods are enabled by solving the dual problem to (5).

The dual problem to (7) is created by introducing dual variables $\boldsymbol{\lambda} \in \mathbb{R}^{n(N-1)}$ for the equality constraints and dual variables $\boldsymbol{\mu} \in \mathbb{R}^{Nn_c}_{\geq 0}$ for the inequality constraints. As shown in [19], the dual problem can explicitly be written as

$$\max_{\boldsymbol{\lambda},\boldsymbol{\mu} \geq 0} -\frac{1}{2}(\mathbf{A}^T\boldsymbol{\lambda} + \mathbf{C}^T\boldsymbol{\mu})^T\mathbf{H}^{-1}(\mathbf{A}^T\boldsymbol{\lambda} + \mathbf{C}^T\boldsymbol{\mu})-$$
$$- \boldsymbol{\lambda}^T\mathbf{b}\bar{x} - \boldsymbol{\mu}^T\mathbf{d}(1-\delta) \quad (8)$$

and we define the minimand in (8) as the dual function for initial condition $\bar{x} \in \mathbb{R}^n$, i.e.,

$$D_N^\delta(\bar{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) := -\frac{1}{2}(\mathbf{A}^T\boldsymbol{\lambda} + \mathbf{C}^T\boldsymbol{\mu})^T\mathbf{H}^{-1}(\mathbf{A}^T\boldsymbol{\lambda} + \mathbf{C}^T\boldsymbol{\mu})-$$
$$- \boldsymbol{\lambda}^T\mathbf{b}\bar{x} - \boldsymbol{\mu}^T\mathbf{d}(1-\delta). \quad (9)$$

The distributed algorithm presented in [19] that solves (7), is a dual accelerated gradient method described by the following global iterations

$$\mathbf{y}^k = -\mathbf{H}^{-1}(\mathbf{A}^T\boldsymbol{\lambda}^k + \mathbf{C}^T\boldsymbol{\mu}^k) \quad (10)$$

$$\bar{\mathbf{y}}^k = \mathbf{y}^k + \frac{k-1}{k+2}(\mathbf{y}^k - \mathbf{y}^{k-1}) \quad (11)$$

$$\boldsymbol{\lambda}^{k+1} = \boldsymbol{\lambda}^k + \frac{k-1}{k+2}(\boldsymbol{\lambda}^k - \boldsymbol{\lambda}^{k-1}) + \frac{1}{L}(\mathbf{A}\bar{\mathbf{y}}^k - \mathbf{b}\bar{x}) \quad (12)$$

$$\boldsymbol{\mu}^{k+1} = \max\left(0, \boldsymbol{\mu}^k + \frac{k-1}{k+2}(\boldsymbol{\mu}^k - \boldsymbol{\mu}^{k-1})+ \right.$$
$$\left. + \frac{1}{L}(\mathbf{C}\bar{\mathbf{y}}^k - \mathbf{d}(1-\delta))\right) \quad (13)$$

where k is the iteration number and $L = \|[\mathbf{A}^T, \mathbf{C}^T]^T\mathbf{H}^{-1}[\mathbf{A}^T, \mathbf{C}^T]\|$, which is the Lipschitz constant to the gradient of the dual function (9). The reader is referred to [19] for details on how to distribute the algorithm (10)-(13).

A. Notation

We define by $\mathbb{N}_{\geq T}$ the set of natural numbers $t \geq T$. The norm $\| \cdot \|$ refers to the Euclidean norm or the induced Euclidean norm unless otherwise is specified and $\langle\cdot,\cdot\rangle$ refers to the inner product in Euclidean space. The norm $\|x\|_T = \sqrt{x^T T x}$. The optimal state and control sequences to (5) for initial value x and constraint tightening δ are denoted $\{z_\tau^*(x,\delta)\}_{\tau=0}^{N-1}$ and $\{v_\tau^*(x,\delta)\}_{\tau=0}^{N-1}$ respectively and the optimal solution to the equivalent problem (7) by $\mathbf{y}^*(x,\delta)$. The state and control sequences for iteration k in (10)-(13) are denoted $\{z_\tau^k(x,\delta)\}_{\tau=0}^{N-1}$ and $\{v_\tau^k(x,\delta)\}_{\tau=0}^{N-1}$ respectively. The initial state and constraint tightening arguments (x,δ) are dropped when no ambiguities can arise.

B. Definitions and assumptions

We adopt the convention that $V_N^\delta(\bar{x}) = \infty$ for states $\bar{x} \in \mathbb{R}^n$ that result in (7) being infeasible. We define by $\mathbb{X}_\infty$ the set

for which (4) is feasible and we define the minimum of the stage-cost ℓ for fixed x as

$$\ell^*(x) := \min_{u\in\mathcal{U}} \ell(x, u) = \frac{1}{2}x^T Q x.$$

Further, κ is the smallest scalar such that $\kappa Q - A^T Q A \succeq 0$. The state sequence resulting from applying $\{v_\tau\}_{\tau=0}^{N-1}$ to (1) is denoted by $\{\xi_\tau\}_{\tau=0}^{N-1}$, i.e.,

$$\xi_{\tau+1} = A\xi_\tau + Bv_\tau, \qquad \xi_0 = \bar{x}. \quad (14)$$

We introduce $\boldsymbol{\xi} = [(\xi_0)^T, \ldots, (\xi_{N-1})^T]^T$ and define the primal cost

$$P_N(\bar{x}, \mathbf{v}) := \begin{cases} \displaystyle\sum_{\tau=0}^{N-1} \ell(\xi_\tau, v_\tau) & \text{if } \boldsymbol{\xi} \in \mathcal{X}^N \text{ and } \mathbf{v} \in \mathcal{U}^N \\ & \text{and (14) holds} \\ \infty & \text{else} \end{cases} \quad (15)$$

where $\mathcal{X}^N$ and $\mathcal{U}^N$ are the state and control constraints for the full horizon. We also introduce the shifted control sequence $\mathbf{v}_s = [(v_1)^T, \ldots, (v_{N-1})^T, 0^T]^T$. We have $P_N(\bar{x}, \mathbf{v}^k) \geq V_N(\bar{x})$ and $P_N(A\bar{x} + Bv_0^k, \mathbf{v}_s^k) \geq V_N(A\bar{x} + Bv_0^k)$ for every algorithm iteration k. We denote by $\{\xi_\tau^k\}_{\tau=0}^{N-1}$ the state sequence that satisfies (14) using controls $\{v_\tau^k\}_{\tau=0}^{N-1}$. The definition of the cost (15) implies

$$P_N(\bar{x}, \mathbf{v}^k) = P_N(A\bar{x} + Bv_0^k, \mathbf{v}_s^k) + \ell(\bar{x}, v_0^k) - \ell^*(A\xi_{N-1}^k) \quad (16)$$

if $v_0^k \in \mathcal{U}$, $\bar{x} \in \mathcal{X}$ and $A\xi_{N-1}^k \in \mathcal{X}$.

III. STOPPING CONDITION

Rather than finding the optimal solution in each time step in the MPC controller, the most important task is to find a control action that gives desirable closed loop properties such as stability, feasibility, and a desired performance. Such properties can sometimes be ensured well before convergence to the optimal solution. To benefit from this observation, a stopping condition is developed that allows the iterations to stop when the desired performance, stability, and feasibility can be guaranteed. Before the stopping condition is introduced, we briefly go through the main ideas below.

A. Main ideas

The distributed nature of the optimization algorithm makes it unsuitable for centralized terminal costs and terminal constraints. Thus, stability and performance need to be ensured without these constructions. We define the following infinite horizon performance for feedback control law ν

$$V_\infty^\nu(\bar{x}) = \sum_{t=0}^\infty \ell(x_t, \nu(x_t)) \quad (17)$$

where $x_{t+1} = Ax_t + B\nu(x_t)$ and $x_0 = \bar{x}$. For a given performance parameter $\alpha \in (0,1]$ and control law ν, it is known (cf. [17], [18], [16]) that the following decrease in the optimal value function

$$V_N^0(x_t) \geq V_N^0(Ax_t + B\nu(x_t)) + \alpha\ell(x, \nu(x_t)) \quad (18)$$

for every $t \in \mathbb{N}_{\geq 0}$ gives stability and closed loop performance according to

$$\alpha V_\infty^\nu(\bar{x}) \leq V_\infty(\bar{x}). \qquad (19)$$

Analysis of the control horizon N needed for an MPC control law without terminal cost and terminal constraints such that (18) holds, is performed in [18], [16] and also in this paper. Once a control horizon N is known such that (18) is guaranteed, the performance result (19) relies on computation of the optimal solution to the MPC optimization problem in every time step. An exact optimal solution cannot be computed and the idea behind this paper is to develop stopping conditions that enable early termination of the optimization algorithm with maintained feasibility, stability, and performance guarantees. The idea behind our stopping condition is to compute a lower bound to $V_N^0(x)$ through the dual function $D_N^0(x, \boldsymbol{\lambda}^k, \boldsymbol{\mu}^k)$ and an upper bound to the next step value function $V_N^0(Ax + Bv_0^k)$ through a feasible solution $P_N(Ax + Bv_0^k, \mathbf{v}_s^k)$. If at iteration k the following test is satisfied

$$D_N^0(\bar{x}, \boldsymbol{\lambda}^k, \boldsymbol{\mu}^k) \geq P_N(A\bar{x} + Bv_0^k, \mathbf{v}_s^k) + \alpha\ell(\bar{x}, v_0^k) \qquad (20)$$

the performance condition (18) holds since

$$V_N^0(\bar{x}) \geq D_N^0(\bar{x}, \boldsymbol{\lambda}^k, \boldsymbol{\mu}^k) \geq P_N(A\bar{x} + Bv_0^k, \mathbf{v}_s^k) + \alpha\ell(\bar{x}, v_0^k)$$
$$\geq V_N^0(A\bar{x} + Bv_0^k) + \alpha\ell(\bar{x}, v_0^k).$$

This implies that stability and the performance result (19) can be guaranteed with finite algorithm iterations k by using control action v_0^k.

The test (20) includes computation of $P_N(A\bar{x} + Bv_0^k, \mathbf{v}_s^k)$ which is a feasible solution to the optimization problem in the following step. A feasible solution cannot be expected with finite number of iterations k for duality-based methods since primal feasibility is only guaranteed in the limit of iterations. Therefore we introduce tightened state and control constraint sets $(1-\delta)\mathcal{X}$, $(1-\delta)\mathcal{U}$ with $\delta \in (0,1)$ and use these in the optimization problem. By generating a state trajectory $\{\xi_\tau^k\}_{\tau=0}^{N-1}$ from the control trajectory $\{v_\tau^k\}_{\tau=0}^{N-1}$ that satisfies the equality constraints (14), we will see that $\{\xi_\tau^k\}_{\tau=0}^{N-1}$ satisfies the original inequality constraints with finite number of iterations. Thus, a primal feasible solution $P_N(A\bar{x} + Bv_0^k, \mathbf{v}_s^k)$ can be generated after a finite number of algorithm iterations k. However, since the optimization now is performed over a tightened constraint set, the dual function value $D_N^\delta(\bar{x}, \boldsymbol{\lambda}, \boldsymbol{\mu})$ is not a lower bound to $V_N^0(\bar{x})$ and cannot be used directly in the test (20) to ensure stability and the performance specified by (19). In the following lemma we show a relation between the dual function value when using the tightened constraint sets and the optimal value function when using the original constraint sets.

Lemma 1: For every $\bar{x} \in \mathbb{R}^n$, $\boldsymbol{\lambda} \in \mathbb{R}^{n(N-1)}$ and $\boldsymbol{\mu} \in \mathbb{R}_{\geq 0}^{Nn_c}$ we have that

$$V_N^0(\bar{x}) \geq D_N^\delta(\bar{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) - \delta\boldsymbol{\mu}^T\mathbf{d}.$$

Proof. From the definition of the dual function (9) we get that

$$D_N^\delta(\bar{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = D_N^0(\bar{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) + \delta\mathbf{d}^T\boldsymbol{\mu}.$$

By weak duality we get

$$V_N^0(\bar{x}) \geq D_N^0(\bar{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = D_N^\delta(\bar{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) - \delta\mathbf{d}^T\boldsymbol{\mu}.$$

This completes the proof. □

The presented lemma enables computation of a lower bound to $V_N^0(\bar{x})$ at algorithm iteration k that depends on $\delta\boldsymbol{\mu}^T\mathbf{d}$. By adapting the amount of constraint tightening δ to satisfy

$$\delta(\boldsymbol{\mu}^k)^T\mathbf{d} \leq \epsilon\ell^*(\bar{x}) \qquad (21)$$

for some $\epsilon > 0$ and use this together with the following test

$$D_N^\delta(\bar{x}, \boldsymbol{\lambda}^k, \boldsymbol{\mu}^k) \geq P_N(A\bar{x} + Bv_0^k, \mathbf{v}_s^k) + \alpha\ell(\bar{x}, v_0^k) \qquad (22)$$

we get from Lemma 1 and if (21) and (22) holds that

$$V_N^0(\bar{x}) \geq D_N^\delta(\bar{x}, \boldsymbol{\lambda}^k, \boldsymbol{\mu}^k) - \delta(\boldsymbol{\mu}^k)^T\mathbf{d}$$
$$\geq P_N(A\bar{x} + Bv_0^k, \mathbf{v}_s^k) + \alpha\ell(\bar{x}, v_0^k) - \epsilon\ell^*(\bar{x})$$
$$\geq V_N^0(A\bar{x} + Bv_0^k) + (\alpha - \epsilon)\ell(\bar{x}, v_0^k).$$

This is condition (18), which guarantees stability and performance specified by (19) if $\alpha > \epsilon$.

B. The stopping condition

Below we state the stopping condition, whereafter parameter settings are discussed.

Algorithm 1: **Stopping condition**

Input: $\bar{x}$
Set: $k = 0$, $l = 0$, $\delta = \delta_{\text{init}}$
Initialize algorithm (10)-(13) with:
$\boldsymbol{\lambda}^0 = \boldsymbol{\lambda}^{-1} = 0, \boldsymbol{\mu}^0 = \boldsymbol{\mu}^{-1} = 0$ and $\mathbf{y}^0 = \mathbf{y}^{-1} = 0$.
Do
    If $D_N^\delta(\bar{x}, \boldsymbol{\lambda}^k, \boldsymbol{\mu}^k) \geq P_N(\bar{x}, \mathbf{v}^k) - \frac{\epsilon}{l+1}\ell^*(\bar{x})$
      or $\delta\mathbf{d}^T\boldsymbol{\mu}^k > \epsilon\ell^*(\bar{x})$
        Set $\delta \leftarrow \delta/2$ // reduce constraint tightening
        Set $l \leftarrow l + 1$
        Set $k = 0$ // reset step size and iteration counter
    End
    Run Δk iterations of (10)-(13)
    Set $k \leftarrow k + \Delta k$
Until $D_N^\delta(\bar{x}, \boldsymbol{\lambda}^k, \boldsymbol{\mu}^k) \geq P_N(A\bar{x} + Bv_0^k, \mathbf{v}_s^k) + \alpha\ell(\bar{x}, v_0^k)$ **and**
    $\delta\mathbf{d}^T\boldsymbol{\mu}^k \leq \epsilon\ell^*(\bar{x})$
Output: v_0^k

In Algorithm 1, four parameters need to be set. The first is the performance parameter $\alpha \in (0,1]$ which guarantees closed loop performance as specified by (19). The larger α, the better performance is guaranteed but a longer control horizon N will be needed to guarantee the specified performance. The second parameter is an initial constraint tightening parameter, which we denote by $\delta_{\text{init}} \in (0,1]$, from which the constraint tightening parameter δ will be adapted (reduced), to satisfy (21). A generic value that always works is $\delta_{\text{init}} = 0.2$, i.e., 20% initial constraint tightening. The third parameter is the relative optimality tolerance $\epsilon > 0$ where $\epsilon < \alpha$. The ϵ must be chosen to satisfy (25). Finally, Δk, which is the number of algorithm iterations between every stopping condition test,

should be set to a positive integer, typically in the range 5 to 20.

Except for the initial condition $\bar{x}$, Algorithm 1 is always identically initialized and follows a deterministic scheme. Thus, for fixed initial condition the same control action is always computed. This implies that Algorithm 1 defines a static feedback control law, which we denote by ν_N. We get the following closed loop dynamics

$$x_{t+1} = Ax_t + B\nu_N(x_t), \qquad x_0 = \bar{x}.$$

The objective of this section is to present a theorem stating that the feedback control law function ν_N satisfies $\mathrm{dom}(\nu_N) \supseteq \mathrm{int}(\mathbb{X}_N^0)$, where

$$\mathbb{X}_N^\delta := \{\bar{x} \in \mathbb{R}^n \mid V_N^\delta(\bar{x}) < \infty \text{ and } Az_{N-1}^*(\bar{x}, 0) \in \mathrm{int}(\mathcal{X})\} \tag{23}$$

which satisfies $\mathbb{X}_N^{\delta_1} \subseteq \mathbb{X}_N^{\delta_2}$ for $\delta_1 > \delta_2$. First, however we state the following definition.

Definition 1: The constant Φ_N is the smallest constant such that the optimal solution $\{z_\tau^*(\bar{x}, 0)\}_{\tau=0}^{N-1}$, $\{v_\tau^*(\bar{x}, 0)\}_{\tau=0}^{N-1}$ to (5) for every $\bar{x} \in \mathbb{X}_N^0$ satisfies

$$\ell^*(z_{N-1}^*(\bar{x}, 0)) \leq \Phi_N \ell(\bar{x}, v_0^*(\bar{x}, 0))$$

for the chosen control horizon N.

The parameter Φ_N is a measure that compares the first and last stage costs in the horizon. In Section IV a method to compute Φ_N is presented.

Remark 1: In [15], [16] an exponential controllability on the stage costs is assumed, i.e., that for $C \geq 1$ and $\sigma \in (0, 1)$ the following holds for $\tau = 0, \ldots, N-1$

$$\ell^*(z_\tau^*(\bar{x}, 0), v_\tau^*(\bar{x}, 0)) \leq C\sigma^\tau \ell(\bar{x}, v_0^*(\bar{x}, 0)).$$

This implies $\Phi_N \leq C\sigma^{N-1}$.

We also need the following lemmas, that are proven in Appendix-A, Appendix-B and Appendix-C respectively, to prove the upcoming theorem.

Lemma 2: Suppose that $\epsilon > 0$ and $\delta \in (0, 1]$. For every $\bar{x} \in \mathbb{X}_N^\delta$ we have for some finite k that

$$D_N^\delta(\bar{x}, \boldsymbol{\lambda}^k, \boldsymbol{\mu}^k) \geq P_N(\bar{x}, \mathbf{v}^k) - \epsilon\ell^*(\bar{x}). \tag{24}$$

Lemma 3: Suppose that $\epsilon > 0$ and $\delta \in (0, 1]$. For every $\bar{x} \in \mathbb{X}_N^\delta$ and algorithm iteration k such that (24) holds we have for $\tau = 0, \ldots, N-1$ that

$$\frac{1}{2}\left\| \begin{bmatrix} \xi_\tau^k(\bar{x}, \delta) \\ v_\tau^k(\bar{x}, \delta) \end{bmatrix} - \begin{bmatrix} z_\tau^*(\bar{x}, 0) \\ v_\tau^*(\bar{x}, 0) \end{bmatrix} \right\|_H^2 \leq \epsilon\ell^*(\bar{x}) + \delta(\boldsymbol{\mu}^k)^T\mathbf{d}$$

where $H = \mathrm{blkdiag}(Q, R)$.

Lemma 4: Suppose that $\epsilon > 0$ and $\delta \in (0, 1]$. For $\bar{x} \in \mathbb{X}_N^0$ but $\bar{x} \notin \mathbb{X}_N^\delta$ we have that $\delta(\boldsymbol{\mu}^k)^T\mathbf{d} > \epsilon\ell^*(\bar{x})$ with finite k.

We are now ready to state the following theorem, which is proven in Appendix-D.

Theorem 1: Assume that $\epsilon > 0$, $\delta_{\mathrm{init}} \in (0, 1]$ and

$$\alpha \leq 1 - \epsilon - \kappa(\sqrt{2\epsilon} + \sqrt{\Phi_N})^2(\sqrt{2\epsilon} + 1)^2. \tag{25}$$

Then the feedback control law ν_N, defined by Algorithm 1, satisfies $\mathrm{dom}(\nu_N) \supseteq \mathrm{int}(\mathbb{X}_N^0)$. Further

$$V_N^0(\bar{x}) \geq V_N^0(A\bar{x} + B\nu_N(\bar{x})) + (\alpha - \epsilon)\ell(\bar{x}, \nu_N(\bar{x})) \tag{26}$$

holds for every $\bar{x} \in \mathrm{dom}(\nu_N)$.

Corollary 1: Suppose that $\alpha \leq 1 - \kappa\Phi_N$ and that $\nu_N^*(\bar{x}) = v_0^*(\bar{x}, 0)$. Then

$$V_N^0(\bar{x}) \geq V_N^0(A\bar{x} + B\nu_N^*(\bar{x})) + \alpha\ell(\bar{x}, \nu_N^*(\bar{x})).$$

holds for every $\bar{x} \in \mathbb{X}_N^0$.

Proof. For every $\bar{x} \in \mathbb{X}_N^0$ we have

$$\begin{aligned} V_N^0(\bar{x}) &= \sum_{\tau=0}^{N-1} \ell(z_\tau^*, u_\tau^*) + \ell(Az_{N-1}^*, 0) - \ell(Az_{N-1}^*, 0) \\ &\geq V_N^0(A\bar{x} + B\nu_N^*(\bar{x})) + \ell(\bar{x}, v_0^*) - \ell(Az_{N-1}^*, 0) \\ &\geq V_N^0(A\bar{x} + B\nu_N^*(\bar{x})) + \ell(\bar{x}, v_0^*) - \kappa\ell(z_{N-1}^*, 0) \\ &\geq V_N^0(A\bar{x} + B\nu_N^*(\bar{x})) + (1 - \kappa\Phi_N)\ell(\bar{x}, v_0^*) \end{aligned}$$

where the first inequality holds since $Az_{N-1}^* \in \mathcal{X}$ by construction of $\mathbb{X}_N^0$, the second due to the definition of κ and the third due to the definition of Φ_N. $\qquad\square$

Remark 2: By setting $\epsilon = 0$ in Theorem 1 we get $\alpha \leq 1 - \kappa\Phi_N$ as in Corollary 1.

C. Feasibility, stability and performance

The following proposition shows one-step feasibility when using the feedback control law ν_N.

Proposition 1: Suppose that α satisfies (25). For every $x_t \in \mathrm{int}(\mathbb{X}_N^0)$ we have that $x_{t+1} = Ax_t + B\nu_N(x_t) \in \mathcal{X}$.

Proof. From Theorem 1 we have that $x_t \in \mathrm{dom}(\nu_N)$ and from Algorithm 1 we have that $P_N(x_{t+1}, \mathbf{v}_s^k) < \infty$ which, by definition, implies that $x_{t+1} \in \mathcal{X}$. $\qquad\square$

The proposition shows that x_{t+1} is feasible if $x_t \in \mathrm{dom}(\nu_N)$. We define the recursively feasible set as the maximal set such that

$$\mathbb{X}_{\mathrm{rf}} = \{x \in \mathcal{X} \mid Ax + B\nu_N(x) \in \mathbb{X}_{\mathrm{rf}}\}$$

In the following theorem we show that $\mathbb{X}_{\mathrm{rf}}$ is the region of attraction and that the control law ν_N achieves a prespecified performance as specified by (17).

Theorem 2: Suppose that $\alpha > \epsilon$ satisfies (25). Then for every initial condition $\bar{x} \in \mathbb{X}_{\mathrm{rf}}$ we have that $\|x_t\| \to 0$ as $t \to \infty$ and that the closed loop performance satisfies

$$(\alpha - \epsilon)V_\infty^{\nu_N}(\bar{x}) \leq V_\infty(\bar{x}). \tag{27}$$

Further, $\mathbb{X}_{\mathrm{rf}}$ is the region of attraction.

Proof. From the definition of $\mathbb{X}_{\mathrm{rf}}$ we know that $\bar{x} = x_0 \in \mathbb{X}_{\mathrm{rf}}$ implies $x_t \in \mathbb{X}_{\mathrm{rf}}$ for all $t \in \mathbb{N}_{\geq 0}$. Since, by construction, $\mathbb{X}_{\mathrm{rf}} \subseteq \mathrm{int}(\mathbb{X}_N^0) \subseteq \mathrm{dom}(\nu_N)$ we have from Theorem 1 that (26) holds for all x_t, $t \in \mathbb{N}_{\geq 0}$. In [18, Proposition 2.2] it was shown, using telescope summation, that (26) implies (27). Further, since the stage cost ℓ satisfies [16, Assumption 5.1] we get from [16, Theorem 5.2] that $\|x_t\| \to 0$ as $t \to \infty$.

What is left to show is that $\mathbb{X}_{\mathrm{rf}}$ is the region of attraction. Denote by $\mathbb{X}_{\mathrm{roa}}$ the region of attraction using ν_N. We have above shown that $\mathbb{X}_{\mathrm{rf}} \subseteq \mathbb{X}_{\mathrm{roa}}$. We next show that $\mathbb{X}_{\mathrm{roa}} \subseteq \mathbb{X}_{\mathrm{rf}}$ by a contradiction argument to conclude that $\mathbb{X}_{\mathrm{rf}} = \mathbb{X}_{\mathrm{roa}}$. Assume that there exist $\bar{x} \in \mathbb{X}_{\mathrm{roa}}$ such that $\bar{x} \notin \mathbb{X}_{\mathrm{rf}}$. If $\bar{x} \in \mathbb{X}_{\mathrm{roa}}$ the closed loop state sequence $\{x_t\}_{t=0}^\infty$ is feasible in every step (and converges to the origin) and consequently

$\{Ax_t + B\nu_N(x_t)\}_{t=0}^{\infty}$ is feasible in every step. This is exactly the requirement to have $\bar{x} \in \mathbb{X}_{\mathrm{rf}}$, which is a contradiction. Thus $\mathbb{X}_{\mathrm{rf}} \subseteq \mathbb{X}_{\mathrm{roa}} \subseteq \mathbb{X}_{\mathrm{rf}}$ which implies that $\mathbb{X}_{\mathrm{rf}} = \mathbb{X}_{\mathrm{roa}}$.

This completes the proof. □

To guarantee a priori that the control law ν_N achieves the performance (27) specified by α, we need to find a control horizon N such that the corresponding controllability parameter Φ_N satisfies (25). This requires the computation of controllability parameter Φ_N which is the topic of the next section.

IV. OFFLINE CONTROLLABILITY VERIFICATION

The stability and performance results in Theorem 2 rely on Definition 1. For the results to be practically meaningful it must be possible to compute Φ_N in Definition 1. In this section we will show that this can be done by solving a mixed integer linear program (MILP). For desired performance specified by α, we get a requirement on the controllability parameter through (25) for Theorem 1 and Theorem 2 to hold. We denote by Φ_α the largest controllability parameter such that Theorem 1 and Theorem 2 holds for the specified α. This parameter is the one that gives equality in (25), i.e., satisfies

$$\alpha = 1 - \epsilon - \kappa(\sqrt{2\epsilon} + \sqrt{\Phi_\alpha})^2(\sqrt{2\epsilon} + 1)^2 \quad (28)$$

for the desired performance α and optimality tolerance ϵ. The parameters α and ϵ must be chosen such that $\Phi_\alpha > 0$. The objective is to find a control horizon N such that the corresponding controllability parameter Φ_N satisfies $\Phi_N \leq \Phi_\alpha$. First we show that for long enough control horizon N there exist a $\Phi_N \leq \Phi_\alpha$.

Lemma 5: Assume that α and ϵ are chosen such that $\Phi_\alpha > 0$ where Φ_α is implicitly defined in (28). Then there exists control horizon N and corresponding controllability parameter $\Phi_N \leq \Phi_\alpha$.

Proof. Since $\mathbb{X}_{\mathrm{rf}}$ is the region of attraction we have $\mathbb{X}_{\mathrm{rf}} \subseteq \mathbb{X}_\infty$. This in turn implies that (7) is feasible for every control horizon $N \in \mathbb{N}_{\geq 1}$ due to the absence of terminal constraints. We have

$$V_N(\bar{x}) = \sum_{\tau=0}^{N-2} \ell(z_\tau^*, v_\tau^*) + \ell(z_{N-1}^*, v_{N-1}^*)$$
$$\geq V_{N-1}(\bar{x}) + \ell(z_{N-1}^*, v_{N-1}^*).$$

Since the pair (A, B) is assumed controllable and since (7) has neither terminal constraints nor terminal cost we have for some finite M that $M \geq V_\infty(\bar{x}) \geq V_N(\bar{x}) \geq V_{N-1}(\bar{x})$. Thus the sequence $\{V_N(\bar{x})\}_{N=0}^{\infty}$ is a bounded monotonic increasing sequence which is well known to be convergent. Thus, for $N \geq \bar{N}$ where $\bar{N}$ is large enough the difference $V_N(\bar{x}) - V_{N-1}(\bar{x})$ is arbitrarily small. Especially $\ell(z_{N-1}^*, v_{N-1}^*) = \ell^*(z_{N-1}^*) \leq V_N(\bar{x}) - V_{N-1}(\bar{x}) \leq \Phi_\alpha \ell(\bar{x}, v_0^*)$ since $\Phi_\alpha > 0$. That is, for long enough control horizon $N \geq \bar{N}$, $\Phi_N \leq \Phi_\alpha$. This completes the proof. □

The preceding Lemma shows that there exists a control horizon N such that $\Phi_N \leq \Phi_\alpha$ if $\Phi_\alpha > 0$ for the chosen performance α and tolerance ϵ. The choice of performance parameter α gives requirements on how ϵ can be chosen to give $\Phi_\alpha > 0$. Larger ϵ requires smaller Φ_α to satisfy (28) which in turn requires longer control horizons N since Φ_N must satisfy $\Phi_N \leq \Phi_\alpha$. In the following section we address the problem of how to compute the control horizon N and corresponding Φ_N such that the desired performance specified by α can be guaranteed.

A. Exact verification of controllability parameter

In the following proposition we introduce an optimization problem that tests if the controllability parameter Φ_N corresponding to control horizon N satisfies $\Phi_N \leq \Phi_\alpha$ for the desired performance specified by α. Before we state the proposition, the following matrices are introduced

$$T = \mathrm{blkdiag}(0, \ldots, 0, -Q, \Phi_\alpha R, 0, \ldots, 0, -R)$$
$$S = \mathrm{blkdiag}(0, \ldots, 0, I, 0, \ldots, 0)$$

where Q and R are the cost matrices for states and inputs and Φ_α is the required controllability parameter for the chosen α. Recalling the partitioning (6) of $\mathbf{y}$ implies that

$$\mathbf{y}^T T \mathbf{y} = \Phi_\alpha v_0^T R v_0 - z_{N-1}^T Q z_{N-1} - v_{N-1}^T R v_{N-1}$$
$$S\mathbf{y} = z_{N-1}$$

Proposition 2: Assume that $\Phi_\alpha > 0$ satisfies (28) for the chosen performance parameter α and optimality tolerance ϵ. Further assume that the control horizon N is such that

$$0 = \min_{\bar{x}} \frac{1}{2} \left(\Phi_\alpha \bar{x}^T Q \bar{x} + \mathbf{y}^T T \mathbf{y} \right) \quad (29)$$
$$\text{s.t. } \bar{x} \in \mathbb{X}_N^0$$
$$\mathbf{y} = \arg\min V_N^0(\bar{x})$$

then $\Phi_N \leq \Phi_\alpha$.

Proof. First we note that $\bar{x} = 0$ gives $\mathbf{y} = 0$ and $\Phi_\alpha \bar{x}^T Q \bar{x} + \mathbf{y}^T T \mathbf{y} = 0$, i.e., we have that 0 is always a feasible solution. Further, (29) implies for every $\bar{x} \in \mathbb{X}_N^0$ that

$$0 \leq \Phi_\alpha \bar{x}^T Q \bar{x} + \mathbf{y}^T T \mathbf{y} = \Phi_\alpha \ell(\bar{x}, v_0^*) - \ell(z_{N-1}^*, v_{N-1}^*)$$
$$= \Phi_\alpha \ell(\bar{x}, v_0^*) - \ell^*(z_{N-1}^*)$$

since $v_{N-1}^* = 0$. This is exactly the condition in Definition 1. Since Φ_N is the smallest such constant, we have $\Phi_N \leq \Phi_\alpha$ for the chosen control horizon N and desired performance α and optimality tolerance ϵ. □

The optimization problem (29) is a bilevel optimization problem with indefinite quadratic cost (see [23] for a survey on bilevel optimization). Such problems are in general NP-hard to solve. The problem can, however, be rewritten as an equivalent MILP as shown in the following proposition which is a straightforward application of [24, Theorem 2].

Proposition 3: Assume that Φ_α satisfies (28) for the chosen performance parameter α and optimality tolerance ϵ. If the

control horizon N is such that the following holds

$$0 = \min \quad -\frac{1}{2}\left(d_x^T \mu^{U1} + d_x^T \mu^{U2} + \mathbf{d}^T \mu^{UL1}\right) \quad (30)$$

$$\text{s.t. } \beta_i^L \in \{0,1\} \ , \ \beta_i^{U1} \in \{0,1\} \ , \ \beta_i^{U2} \in \{0,1\}$$

Upper level

Primal and dual feasibility
$$C_x \bar{x} - d_x - s^x = 0$$
$$s^x \leq 0 \ , \ \mu^{U1} \geq 0$$
$$C_x AS\mathbf{y} - d_x - s^z = 0$$
$$s^z \leq 0 \ , \ \mu^{U2} \geq 0$$

Stationarity
$$\Phi_\alpha Q\bar{x} + (C_x)^T \mu^{U1} - \mathbf{b}^T \lambda^{UL2} = 0$$
$$T\mathbf{y} + H^T \lambda^{UL1} + \mathbf{A}^T \lambda^{UL2} + \mathbf{C}^T \mu^{UL1}$$
$$+ (C_x AS)^T \mu^{U2} = 0$$
$$\mathbf{A}\lambda^{UL1} = 0$$
$$\mathbf{C}\lambda^{UL1} - \mu^{UL2} = 0$$

Complementarity
$$\beta_i^L = 1 \Rightarrow \mu_i^{UL2} = 0 \ , \ \beta_i^L = 0 \Rightarrow \mu_i^{UL1} = 0$$
$$\beta_i^{U1} = 1 \Rightarrow s_i^x = 0 \ , \ \beta_i^{U1} = 0 \Rightarrow \mu_i^{U1} = 0$$
$$\beta_i^{U2} = 1 \Rightarrow s_i^z = 0 \ , \ \beta_i^{U2} = 0 \Rightarrow \mu_i^{U2} = 0$$

Lower level

Primal and dual feasibility
$$\mathbf{A}\mathbf{y} - \mathbf{b}\bar{x} = 0$$
$$\mathbf{C}\mathbf{y} - \mathbf{d} - s = 0$$
$$s \leq 0 \ , \ \mu^L \geq 0$$

Stationarity
$$H\mathbf{y} + \mathbf{A}^T \lambda^L + \mathbf{C}^T \mu^L = 0$$

Complementarity
$$\beta_i^L = 1 \Rightarrow s_i = 0 \ , \ \beta_i^L = 0 \Rightarrow \mu_i^L = 0$$

where all β, μ, λ, s and $\bar{x}, \mathbf{y}$ are decision variables, then $\Phi_\alpha \geq \Phi_N$.

Proof. The set $\mathbb{X}_N^0$ can equivalently be written as

$$\mathbb{X}_N^0 = \{x \in \mathbb{R}^n \mid \mathbf{A}\mathbf{y}^*(x,0) = \mathbf{b}x, \mathbf{C}\mathbf{y}^*(x,0) \leq \mathbf{d},$$
$$C_x AS\mathbf{y}^*(x,0) \leq d_x, C_x x \leq d_x\}. \quad (31)$$

We express the set $\mathbb{X}_N^0$ in (29) using (31). The equivalence between the optimization problems (30) and (29) is established in [24, Theorem 2]. The remaining parts of the proposition follow by applying Proposition 2. □

The transformation from (29) to (30) is done by expressing the lower level optimization problem in (29) by its sufficient and necessary KKT conditions to get a single level indefinite quadratic program with complementarity constraints. The resulting indefinite quadratic program with complementarity constraints can in turn be cast as a MILP to get (30).

Remark 3: Although MILP problems are NP-hard, there are efficient solvers available such as CPLEX and GUROBI. There are also solvers available for solving the bilevel optimization problem (29) directly, e.g., the function *solvebilevel* in YALMIP, [25].

If the chosen control horizon N is not long enough for $\Phi_N \leq \Phi_\alpha$, different heuristics can be used to choose a new longer horizon to be verified. One heuristic is to assume exponential controllability as in Remark 1, i.e., that there exist constants $C \geq 1$ and $\sigma \in (0,1)$ such that

$$C\sigma^\tau \ell(\bar{x}, v_0^k) \geq \ell(z_\tau^k, v_\tau^k) \quad (32)$$

for all $\tau = 0, \ldots, N-1$. The C and σ-parameters should be determined using the optimal solution $\mathbf{y}$ to (7) for the x that minimized (30) in the previous test. Under the assumption that (32) holds as N increases, a new guess on the control horizon N can be computed by finding the smallest N such that $C\sigma^{N-1} \leq \Phi_\alpha$.

B. Controllability parameter estimation

The test in Proposition 3 verifies if the control horizon N is long enough for the controllability assumption to hold for the required controllability parameter Φ_α. Thus, an initial guess on the control horizon is needed. A guaranteed lower bound can easily be computed by solving (7) for a variety of initial conditions $\bar{x}$ and compute the worst controllability parameter, denoted by $\hat{\Phi}_N$, for these sample points. If the estimated controllability parameter $\hat{\Phi}_N \geq \Phi_\alpha$, we know that the control horizon need to be increased for (30) to hold. If instead $\hat{\Phi}_N \leq \Phi_\alpha$ the control horizon N might serve as a good initial guess to be verified by (30).

Remark 4: For large systems, (30) may be too complex to verify the desired performance. In such cases, the heuristic method mentioned above can be used in conjunction with an adaptive horizon scheme. The adaptive scheme keeps the horizon fixed for all time-steps until the controllability assumption does not hold. Then, the control horizon is increased to satisfy the assumption and kept at the new level until the controllability assumption does not hold again. Eventually the control horizon will be large enough for $\Phi_N \leq \Phi_\alpha$ and the horizon need not be increased again.

V. NUMERICAL EXAMPLE

We evaluate the efficiency of the proposed distributed feedback control law ν_N by applying it to a randomly generated dynamical system with sparsity structure that is specified in [26, Supplement A.1]. The random dynamics matrix is scaled such that the magnitude of the largest eigenvalue is 1.1. The system has 3 subsystems with 5 states and 1 input each. All state variables are upper and lower bounded by random numbers in the intervals $[0.5, 1.5]$ and $[-0.15, -0.05]$ respectively and all input variables are upper and lower bounded by random numbers in the intervals $[0.5, 1.5]$ and $[-0.5, -1.5]$ respectively. The stage cost is chosen to be

$$\ell_i(x_i, u_i) = x_i^T x_i + u_i^T u_i$$

for $i = 1, 2, 3$. The suboptimality parameter is chosen $\alpha = 0.01$. According to Theorem 1, to quantify the control horizon $N(\alpha)$, the optimality tolerance ϵ must be chosen and κ computed, where κ is the smallest constant such that $\kappa Q \succeq A^T Q A$. We get $\kappa = 1.22$ and choose $\epsilon = 0.005$. Using (25), we get $\Phi_{N(0.01)} \leq 0.51$. Verification by solving the MILP in (30) gives that the smallest control horizon $N(0.01)$ that satisfies $\Phi_{N(0.01)} \leq 0.51$ is $N(0.01) = 6$.

Table I presents the results. The first column specifies the stopping condition used, "stop. cond." for the stopping

TABLE I
EXPERIMENTAL RESULTS FOR DIFFERENT PERFORMANCE REQUIREMENTS
α AND DIFFERENT INITIAL CONSTRAINT TIGHTENINGS δ_{init}.

Algorithm comparison, $\alpha = 0.01$, $N = 6$

condition	ϵ	δ_{init}	avg. # iters	max # iters	avg. δ
stop. cond.	0.005	0.001	288.3	506	0.001
stop. cond.	0.005	0.01	151.5	260	0.01
stop. cond.	0.005	0.05	73.7	237	0.049
stop. cond.	0.005	0.1	70.7	236	0.057
stop. cond.	0.005	0.2	72.8	236	0.060
stop. cond.	0.005	0.5	69.2	234	0.076
opt. cond.	0.005	0.001	324.5	506	0.001
opt. cond.	0.005	0.01	171.5	260	0.01

condition presented in Algorithm 1 and "opt. cond." for a optimality conditions. The second column specifies the duality gap tolerance ϵ and the third column specifies the initial constraint tightening δ_{init} for the stopping condition and the relative accuracy requirement for the constraints when using optimality conditions.

Columns four, five and six contain the simulation results. The results are obtained by simulating the system with 1000 randomly chosen initial conditions that are drawn from a uniform distribution on $\mathcal{X}$. Column four and five contain the mean and max numbers of iterations needed and column six presents the average constraint tightening δ used at termination of Algorithm 1.

We see that the adaptive constraint tightening approach gives considerably less iterations for a larger initial tightening. However, for more than 10% initial constraint tightening ($\delta_{\text{init}} = 0.1$), the number of iterations is not significantly affected. It is remarkable to note that 50% initial constraint tightening ($\delta_{\text{init}} = 0.5$) is as efficient as, e.g., 5% ($\delta_{\text{init}} = 0.05$) considering that more reductions in the constraint tightening need to be performed. This indicates early detection of infeasibility. We also note that for a suitable choice of initial constraint tightening, the average number of iterations is reduced significantly.

VI. CONCLUSIONS

We have equipped the duality-based distributed optimization algorithm in [19], when used in a DMPC context, with a stopping condition that guarantees feasibility of the optimization problem and stability and a prespecified performance of the closed-loop system. A numerical example is provided that shows that the stopping condition can reduce significantly the number of iterations needed to achieve these properties.

VII. ACKNOWLEDGMENTS

REFERENCES

[1] A. Richards and J. How, "Robust distributed model predictive control," *International Journal of Control*, vol. 80, no. 9, pp. 1517–1531, Sep. 2007.

[2] W. Dunbar, "Distributed receding horizon control of dynamically coupled nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 52, no. 7, pp. 1249–1263, Jul. 2007.

[3] R. Hermans, M. Lazar, and A. Jokic, "Almost decentralized lyapunov-based nonlinear model predictive control," in *Proceedings of the 2010 American Control Conference*, Baltimore, July 2010, pp. 3932 – 3938.

[4] D. Jia and B. Krogh, "Distributed model predictive control," in *Proceedings of the 2001 American Control Conference*, Arlington, VA, Jun. 2001, pp. 2767–2772.

[5] E. Camponogara, D. Jia, B. Krogh, and S. Talukdar, "Distributed model predictive control," *IEEE Control Systems Magazine*, vol. 22, no. 1, pp. 44–52, Feb. 2002.

[6] A. Venkat, I. Hiskens, J. Rawlings, and S. Wright, "Distributed MPC strategies with application to power system automatic generation control," *IEEE Transactions on Control Systems Technology*, vol. 16, no. 6, pp. 1192–1206, Nov. 2008.

[7] R. Negenborn, B. De Schutter, and J. Hellendoorn, "Multi-agent model predictive control for transportation networks: Serial versus parallel schemes," *Engineering Applications of Artificial Intelligence*, vol. 21, no. 3, pp. 353–366, Apr. 2008.

[8] Y. Wakasa, M. Arakawa, K. Tanaka, and T. Akashi, "Decentralized model predictive control via dual decomposition," in *Proceedings of the 47th IEEE Conference on Decision and Control*, Cancun, Mexico, Dec. 2008, pp. 381 – 386.

[9] D. Doan, T. Keviczky, and B. De Schutter, "An improved distributed version of Han's method for distributed MPC of canal systems," in *12th symposium on Large Scale Systems: Theory and Applications*, Villeneuve d'Ascq, France, July 2010.

[10] I. Necoara and J. Suykens, "Application of a smoothing technique to decomposition in convex optimization," *IEEE Transactions on Automatic Control*, vol. 53, no. 11, pp. 2674 –2679, Dec. 2008.

[11] Y. Nesterov, "Smooth minimization of non-smooth functions," *Math. Program.*, vol. 103, no. 1, pp. 127–152, May 2005.

[12] D. Doan, T. Keviczky, I. Necoara, M. Diehl, and B. De Schutter, "A distributed version of Han's method for DMPC using local communications only," *Control Engineering and Applied Informatics*, vol. 11, no. 3, pp. 6–15, 2009.

[13] D. Q. Mayne, J. B. Rawlings, C. V. Rao, and P. O. M. Scokaert, "Constrained model predictive control: Stability and optimality," *Automatica*, vol. 36, no. 6, pp. 789 – 814, Jun. 2000.

[14] J. Rawlings and D. Mayne, *Model Predictive Control: Theory and Design*. Nob Hill Publishing, 2009.

[15] G. Grimm, M. J. Messina, S. E. Tuna, and A. R. Teel, "Model predictive control: for want of a local control Lyapunov function, all is not lost," *IEEE Transactions on Automatic Control*, vol. 50, no. 5, pp. 546–558, 2005.

[16] L. Grüne, "Analysis and design of unconstrained nonlinear MPC schemes for finite and infinite dimensional systems," *SIAM Journal on Control and Optimization*, vol. 48, no. 8, pp. 4938–4962, 2009.

[17] B. Lincoln and A. Rantzer, "Relaxing dynamic programming," *IEEE Transactions on Automatic Control*, vol. 51, no. 8, pp. 1249–1260, Aug. 2006.

[18] L. Grüne and A. Rantzer, "On the infinite horizon performance of receding horizon controllers," *IEEE Transactions on Automatic Control*, vol. 53, no. 9, pp. 2100–2111, Oct. 2008.

[19] P. Giselsson, M. D. Doan, T. Keviczky, B. De Schutter, and A. Rantzer, "Accelerated gradient methods and dual decomposition in distributed model predictive control," *Automatica*, submitted.

[20] S. Boyd, N. Parikh, E. Chu, B. Peleato, and J. Eckstein, "Distributed optimization and statistical learning via the alternating direction method of multipliers," *Foundations and Trends in Machine Learning*, vol. 3, no. 1, pp. 1–122, 2011.

[21] J. R. Gossner, B. Kouvaritakis, and J. A. Rossiter, "Stable generalized predictive control with constraints and bounded disturbances," *Automatica*, vol. 33, no. 4, pp. 551–568, Apr. 1997. [Online]. Available: http://dx.doi.org/10.1016/S0005-1098(96)00214-2

[22] M. Doan, T. Keviczky, and B. De Schutter, "A dual decomposition-based optimization method with guaranteed primal feasibility for hierarchical MPC problems," in *Proceedings of the 18th IFAC World Congress*, Milan, Italy, Aug.–Sep. 2011, pp. 392–397.

[23] B. Colson, P. Marcotte, and G. Savard, "Bilevel programming: A survey," *4OR: A Quarterly Journal of Operations Research*, vol. 3, no. 2, pp. 87–107, 2005.

[24] C. Jones and M. Morari, "Approximate Explicit MPC using Bilevel Optimization," in *European Control Conference*, Budapest, Hungary, Aug. 2009.

[25] J. Löfberg, "Yalmip : A toolbox for modeling and optimization in MATLAB," in *Proceedings of the CACSD Conference*, Taipei, Taiwan, 2004.

[26] P. Giselsson, "Gradient-based distributed model predictive control," Ph.D. dissertation, Department of Automatic Control, Lund University, Sweden, Nov. 2012.

[27] Y. Nesterov, *Introductory Lectures on Convex Optimization: A Basic Course (Applied Optimization)*, 1st ed. Springer Netherlands, 2003.

[28] S. Boyd and L. Vandenberghe, *Convex Optimization*. New York, NY, USA: Cambridge University Press, 2004.

[29] A. Beck and M. Teboulle, "A fast iterative shrinkage-thresholding algorithm for linear inverse problems," *SIAM J. Imaging Sciences*, vol. 2, no. 1, pp. 183–202, Oct 2009.

[30] A. Nedić and A. Ozdaglar, "Approximate primal solutions and rate analysis for dual subgradient methods," *SIAM J. on Optimization*, vol. 19, no. 4, pp. 1757–1780, Feb. 2009.

A. Proof for Lemma 2

We divide the proof into two parts, the first for $\bar{x} = 0$ and the second for $\bar{x} \neq 0$. For $\bar{x} = 0$ we have at iteration $k = 0$ that $\mathbf{y}^0 = 0$ which is the optimal solution. Hence (24) holds for $k = 0$ since all terms are 0 and $0 = A\xi^0_{N-1} \in \mathcal{X}$.

Next, we show the result for $\bar{x} \neq 0$. Whenever (7) is feasible we have convergence in primal variables [19, Theorem 1]. This together with the linear relation through which ξ is defined (14) gives $\xi^k_\tau \to z^*_\tau$ for $\tau = 0, \ldots, N-1$ as $k \to \infty$. We have $z^*_\tau \in (1-\delta)\mathcal{X}$ and since $(1-\delta)\mathcal{X} \subset \mathcal{X}$ for every $\delta \in (0,1]$ this implies that there exists finite k^x_0 such that $\xi^k_\tau \in \mathcal{X}$ for all $k \geq k^x_0$. Equivalent convergence reasoning holds for v^k_τ. Together this implies that there exists finite k^P_0 such that $P_N(\bar{x}, \mathbf{v}^k) < \infty$ and that $P_N(\bar{x}, \mathbf{v}^k) \to V^\delta_N(\bar{x})$ for all $k \geq k^P_0$. Together with convergence in dual function value [19, Theorem 1] gives that

$$D^\delta_N(\bar{x}, \boldsymbol{\lambda}^k, \boldsymbol{\mu}^k) \geq P_N(\bar{x}, \mathbf{v}^k) - \epsilon\ell^*(\bar{x})$$

holds with finite k since $\ell^*(\bar{x}) > 0$ and $\epsilon > 0$. This concludes the proof. $\square$

B. Proof for Lemma 3

We introduce $\mathbf{y}^k = [(\boldsymbol{\xi}^k(\bar{x},\delta))^T (\mathbf{v}^k(\bar{x},\delta))^T]^T$, where $\boldsymbol{\xi}^k(\bar{x},\delta)$ and $\mathbf{v}^k(\bar{x},\delta)$ satisfies the dynamic equations (14). Whenever (24) holds we have that $\xi^k_\tau(\bar{x},\delta) \in \mathcal{X}$ and $v^k_\tau(\bar{x},\delta) \in \mathcal{U}$ for $\tau = 0, \ldots, N-1$. We also introduce $\mathbf{y}^* = [(z^*(\bar{x},0))^T (\mathbf{v}^*(\bar{x},0))^T]^T$. This implies

$$\frac{1}{2}(\mathbf{y}^k - \mathbf{y}^*)^T \mathbf{H}(\mathbf{y}^k - \mathbf{y}^*) =$$
$$= \frac{1}{2}(\mathbf{y}^k)^T \mathbf{H}\mathbf{y}^k - \frac{1}{2}(\mathbf{y}^*)^T \mathbf{H}\mathbf{y}^* - \langle H\mathbf{y}^*, \mathbf{y}^k - \mathbf{y}^* \rangle$$
$$\leq P_N(\bar{x}, \mathbf{v}^k) - V^0_N(\bar{x}) \leq D^\delta_N(\bar{x}, \boldsymbol{\lambda}^k, \boldsymbol{\mu}^k) + \epsilon\ell^*(\bar{x}) - V^0_N(\bar{x})$$
$$\leq \delta(\boldsymbol{\mu}^k)^T \mathbf{d} + \epsilon\ell^*(\bar{x})$$

where the first inequality comes from the first order optimality condition [27, Theorem 2.2.5] and by definition of V^0_N and P_N. The second inequality is due to (24) and the last inequality follows from Lemma 1. Further, since $\mathbf{H} = \text{blkdiag}(Q, \ldots, Q, R, \ldots, R)$ we have for $\tau = 0, \ldots, N-1$ that

$$\frac{1}{2} \left\| \begin{bmatrix} \xi^k_\tau(\bar{x},\delta) \\ v^k_\tau(\bar{x},\delta) \end{bmatrix} - \begin{bmatrix} z^*_\tau(\bar{x},0) \\ v^*_\tau(\bar{x},0) \end{bmatrix} \right\|^2_H \leq \frac{1}{2}(\mathbf{y}^k - \mathbf{y}^*)^T \mathbf{H}(\mathbf{y}^k - \mathbf{y}^*)$$
$$\leq \delta(\boldsymbol{\mu}^k)^T \mathbf{d} + \epsilon\ell^*(\bar{x})$$

where $H = \text{blkdiag}(Q, R)$, whenever (24) holds. This completes the proof. $\square$

C. Proof for Lemma 4

Since $x \in \mathbb{X}^0_N$ but $x \notin \mathbb{X}^\delta_N$ we have that $V^0_N(\bar{x}) < \infty$ and $V^\delta_N(\bar{x}) = \infty$. Further, from the strong theorem of alternatives [28, Section 5.8.2] we know that since $V^\delta_N(\bar{x}) = \infty$ for the current constraint tightening δ the dual problem is unbounded. Hence there exist $\boldsymbol{\lambda}_f$, $\boldsymbol{\mu}_f$ such that

$$\delta\boldsymbol{\mu}^T_f \mathbf{d} \geq D^\delta_N(\bar{x}, \boldsymbol{\lambda}_f, \boldsymbol{\mu}_f) - V^0_N(\bar{x}) \geq 2\epsilon\ell^*(\bar{x}) \qquad (33)$$

where Lemma 1 is used in the first inequality. Further, the convergence rate in [29, Theorem 4.4] for algorithm (10)-(13) is

$$D^\delta_N(\bar{x}, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*) - D^\delta_N(\bar{x}, \boldsymbol{\lambda}^k, \boldsymbol{\mu}^k) \leq \frac{2L}{(k+1)^2} \left\| \begin{bmatrix} \boldsymbol{\lambda}^* \\ \boldsymbol{\mu}^* \end{bmatrix} - \begin{bmatrix} \boldsymbol{\lambda}^0 \\ \boldsymbol{\mu}^0 \end{bmatrix} \right\|^2.$$

By inspecting the proof to [29, Theorem 4.4] (and [29, Lemma 2.3, Lemma 4.1]) it is concluded that the optimal point $\boldsymbol{\lambda}^*, \boldsymbol{\mu}^*$ can be changed to any feasible point $\boldsymbol{\lambda}_f, \boldsymbol{\mu}_f$ and the convergence result still holds, i.e.,

$$D^\delta_N(\bar{x}, \boldsymbol{\lambda}_f, \boldsymbol{\mu}_f) - D^\delta_N(\bar{x}, \boldsymbol{\lambda}^k, \boldsymbol{\mu}^k) \leq \frac{2L}{(k+1)^2} \left\| \begin{bmatrix} \boldsymbol{\lambda}_f \\ \boldsymbol{\mu}_f \end{bmatrix} - \begin{bmatrix} \boldsymbol{\lambda}^0 \\ \boldsymbol{\mu}^0 \end{bmatrix} \right\|^2.$$

That is, there exists a feasible pair $(\boldsymbol{\lambda}_f, \boldsymbol{\mu}_f)$ such that with finite k we have

$$D^\delta_N(\bar{x}, \boldsymbol{\lambda}^k, \boldsymbol{\mu}^k) > D^\delta_N(\bar{x}, \boldsymbol{\lambda}_f, \boldsymbol{\mu}_f) - \epsilon\ell^*(\bar{x}). \qquad (34)$$

This implies

$$\delta\mathbf{d}^T \boldsymbol{\mu}^k \geq D^\delta_N(\bar{x}, \boldsymbol{\lambda}^k, \boldsymbol{\mu}^k) - V^0_N(\bar{x})$$
$$> D^\delta_N(\bar{x}, \boldsymbol{\lambda}_f, \boldsymbol{\mu}_f) - V^0_N(\bar{x}) - \epsilon\ell^*(\bar{x}) \geq \epsilon\ell^*(\bar{x})$$

where Lemma 1 is used in the first inequality, (34) in the second inequality and (33) in the final inequality. This completes the proof. $\square$

D. Proof for Theorem 1

To prove the assertion we need to show that the do loop will exit for every $\bar{x} \in \text{int}(\mathbb{X}^0_N)$. For every point $\bar{x} \in \text{int}(\mathbb{X}^0_N)$ there exists $\bar{\delta} \in (0,1)$ such that $\frac{\bar{x}}{1-\bar{\delta}} \in \text{int}(\mathbb{X}^0_N)$. Since $\text{int}(\mathbb{X}^0_N) \subseteq \mathbb{X}^0_N$, we have that $V^0_N(\frac{\bar{x}}{1-\bar{\delta}}) < \infty$ and the optimal solution $\mathbf{y}^*(\frac{\bar{x}}{1-\bar{\delta}}, 0)$ satisfies $\mathbf{A}\mathbf{y}^*(\frac{\bar{x}}{1-\bar{\delta}}, 0) = \mathbf{b}\frac{\bar{x}}{1-\bar{\delta}}$ and $\mathbf{C}\mathbf{y}^*(\frac{\bar{x}}{1-\bar{\delta}}, 0) \leq \mathbf{d}$. We create the following vector

$$\bar{\mathbf{y}}(\bar{x}) := (1 - \bar{\delta})\mathbf{y}^*(\frac{\bar{x}}{1-\bar{\delta}}, 0) \qquad (35)$$

which satisfies

$$\mathbf{A}\bar{\mathbf{y}}(\bar{x}) = \mathbf{A}\mathbf{y}^*(\frac{\bar{x}}{1-\bar{\delta}}, 0)(1 - \bar{\delta}) = \mathbf{b}\bar{x}\frac{1-\bar{\delta}}{1-\bar{\delta}} = \mathbf{b}\bar{x} \qquad (36)$$

$$\mathbf{C}\bar{\mathbf{y}}(\bar{x}) = \mathbf{C}\mathbf{y}^*(\frac{\bar{x}}{1-\bar{\delta}}, 0)(1 - \bar{\delta}) \leq \mathbf{d}(1 - \bar{\delta}). \qquad (37)$$

Hence, by definition (23) of $\mathbb{X}^\delta_N$ we conclude that for every $\bar{x} \in \text{int}(\mathbb{X}^0_N)$ there exist $\bar{\delta} \in (0,1)$ such that $\bar{x} \in \mathbb{X}^{\bar{\delta}}_N$. This implies that for every $\bar{x} \in \text{int}(\mathbb{X}^0_N)$ we have that either $\bar{x} \in \mathbb{X}^\delta_N$ for the current constraint tightening $\delta \in (0,1)$ or $\bar{x} \notin \mathbb{X}^\delta_N$ but $\bar{x} \in \mathbb{X}^0_N$. Thus, from Lemma 2 and Lemma 4 we conclude

that either the do loop is terminated or δ is reduced and l is increased for every $\bar{x} \in \mathrm{int}(\mathbb{X}_N^0)$ with finite number of algorithm iterations k.

To guarantee that the do loop will terminate for every $\bar{x} \in \mathrm{int}(\mathbb{X}_N^0)$, we need to show that the conditions in the do loop will hold for small enough δ and with finite k. That is, we need to show that the following two conditions will hold.

1) For small enough δ, i.e., large enough l, we have that

$$\delta(\boldsymbol{\mu}^k)^T\mathbf{d} \leq \epsilon\ell^*(\bar{x}) \tag{38}$$

where $\delta = 2^{-l}\delta_{\mathrm{init}}$ holds for every algorithm iteration k.

2) For small enough δ, i.e., large enough l, the condition

$$D_N^\delta(\bar{x}, \boldsymbol{\lambda}^k, \boldsymbol{\mu}^k) \geq P_N(A\bar{x}+Bv_0^k, \mathbf{v}_s^k)+\alpha\ell(\bar{x}, v_0^k) \tag{39}$$

with α satisfying (25) holds with finite k whenever

$$D_N^\delta(\bar{x}, \boldsymbol{\lambda}^k, \boldsymbol{\mu}^k) \geq P_N(\bar{x}, \mathbf{v}^k) + \frac{\epsilon}{l+1}\ell(\bar{x}, v_0^k) \tag{40}$$

holds.

We start by showing argument 1. From the convergence rate of the algorithm [19] it follows that there exists $\underline{D} > -\infty$ such that $D_N^\delta(\bar{x}, \boldsymbol{\lambda}^k, \boldsymbol{\mu}^k) \geq \underline{D}$ for every algorithm iteration $k \geq 0$. This is used below where we extend the result from [30, Lemma 1] to handle the presence of equality constraints. For algorithm iteration $k \geq 0$, $\bar{x} \in \mathrm{int}(\mathbb{X}_N^0)$ and $\delta \leq \bar{\delta}/2$ we have

$$\underline{D} \leq D_N^\delta(\bar{x}, \boldsymbol{\lambda}^k, \boldsymbol{\mu}^k)$$
$$= \inf_{\mathbf{y}} \frac{1}{2}\mathbf{y}^T\mathbf{H}\mathbf{y} + (\boldsymbol{\lambda}^k)^T(\mathbf{A}\mathbf{y} - \mathbf{b}\bar{x})+$$
$$+ (\boldsymbol{\mu}^k)^T(\mathbf{C}\mathbf{y} - (1-\delta)\mathbf{d})$$
$$\leq \frac{1}{2}(\bar{\mathbf{y}}(\bar{x}))^T\mathbf{H}\bar{\mathbf{y}}(\bar{x}) + (\boldsymbol{\lambda}^k)^T(\mathbf{A}\bar{\mathbf{y}}(\bar{x}) - \mathbf{b}\bar{x})+$$
$$+ (\boldsymbol{\mu}^k)^T(\mathbf{C}\bar{\mathbf{y}}(\bar{x}) - (1-\delta)\mathbf{d})$$
$$\leq (1-\bar{\delta})^2 V_N^0(\frac{\bar{x}}{1-\bar{\delta}}) + (\boldsymbol{\mu}^k)^T(\mathbf{C}\bar{\mathbf{y}}(\bar{x}) - (1-\bar{\delta})\mathbf{d})+$$
$$+ (\boldsymbol{\mu}^k)^T\mathbf{d}(\delta - \bar{\delta})$$
$$\leq V_N^0(\frac{\bar{x}}{1-\bar{\delta}}) + (\boldsymbol{\mu}^k)^T\mathbf{d}(\delta - \bar{\delta})$$
$$\leq V_N^0(\frac{\bar{x}}{1-\bar{\delta}}) - \frac{1}{2}(\boldsymbol{\mu}^k)^T\mathbf{d}\bar{\delta}$$

where the equality is by definition, the second inequality holds since any vector $\bar{\mathbf{y}}(\bar{x})$ is gives larger value than the infimum, the third and fourth inequalities are due to (35), (36) and (37) and since $(1-\bar{\delta}) \in (0,1)$ and the final inequality holds since $\delta \leq \bar{\delta}/2$. This implies that

$$(\boldsymbol{\mu}^k)^T\mathbf{d} \leq \frac{2(V_N^0(\frac{\bar{x}}{1-\bar{\delta}}) - \underline{D})}{\bar{\delta}}$$

which is finite. We denote by l_d the smallest l such that $\bar{\delta} \geq 2^{-l_d}\delta_{\mathrm{init}}$. Since $\delta = 2^{-l}\delta_{\mathrm{init}}$ this implies that

$$\delta(\boldsymbol{\mu}^k)^T\mathbf{d} \leq \delta\frac{2(V_N^0(\frac{\bar{x}}{1-\bar{\delta}}) - \underline{D})}{\bar{\delta}} \leq 2^{-l}\delta_{\mathrm{init}}\frac{2(V_N^0(\frac{\bar{x}}{1-\bar{\delta}}) - \underline{D})}{2^{-l_d}\delta_{\mathrm{init}}}$$
$$\leq 2^{-l+l_d+1}(V_N^0(\frac{\bar{x}}{1-\bar{\delta}}) - \underline{D}) \to 0 \tag{41}$$

as $l \to \infty$. Especially, with finite l we have that (38) holds for every algorithm iteration k. This proves argument 1.

Next we prove argument 2. We start by showing for large enough but finite l that $P^N(A\bar{x} + B\nu_N(\bar{x}), \mathbf{v}_s^k)$ is finite whenever (40) holds. From the definition of P_N and $\mathbf{v}_s^k$ we have that $P^N(A\bar{x}+B\nu_N(\bar{x}), \mathbf{v}_s^k)$ is finite whenever $P_N(\bar{x}, \mathbf{v}_s^k)$ is finite and if $A\xi_{N-1}^k(\bar{x}, \delta) \in \mathcal{X}$. For algorithm iteration k such that (40) holds we have

$$\|A(\xi_{N-1}^k(\bar{x}, \delta) - z_{N-1}^*(\bar{x}, 0))\|^2 \leq$$
$$\leq \frac{\|A\|^2}{\lambda_{\min}(H)}\|\xi_{N-1}^k(\bar{x}, \delta) - z_{N-1}^*(\bar{x}, 0)\|_H^2$$
$$\leq \frac{2\|A\|^2}{\lambda_{\min}(H)}(\delta(\boldsymbol{\mu}^k)^T\mathbf{d} + \frac{\epsilon}{l+1}\ell^*(\bar{x}))$$
$$\leq \frac{2\|A\|^2}{\lambda_{\min}(H)}\left(2^{-l+l_d+1}(V_N^0(\frac{\bar{x}}{1-\bar{\delta}}) - \underline{D}) + \frac{\epsilon}{l+1}\ell^*(\bar{x})\right) \to 0 \tag{42}$$

as $l \to \infty$ where $H = \mathrm{blkdiag}(Q, R)$ and the smallest eigenvalue $\lambda_{\min}(H) > 0$ since H is positive definite. The first inequality follows from Cauchy-Schwarz inequality and Courant-Fischer-Weyl min-max principle, the second inequality comes from Lemma 3 and the third comes from (41). By definition of $\mathbb{X}_N^\delta$ we have $Az_{N-1}^*(\bar{x}, 0) \in \mathrm{int}(\mathcal{X})$ which through (42) implies that $A\xi_{N-1}^k(\bar{x}, \delta) \in \mathcal{X}$ for some large enough by finite l, i.e., small enough δ, and for algorithm iteration k such that (40) holds.

What is left to show is that (39) holds for every $\alpha \leq 1 - 2\epsilon - \kappa(\sqrt{2\epsilon} + \sqrt{\Phi_N})^2(\sqrt{2\epsilon} + 1)^2$ for large enough but finite l whenever (40) holds. From Lemma 3 and (41) we know for large enough l and any algorithm iteration k such that (40) holds that

$$\frac{1}{2}\left\|\begin{bmatrix}\xi_\tau^k \\ v_\tau^k\end{bmatrix} - \begin{bmatrix}z_\tau^* \\ v_\tau^*\end{bmatrix}\right\|_H^2 \leq \delta(\boldsymbol{\mu}^k)^T\mathbf{d} + \frac{\epsilon}{l+1}\ell^*(\bar{x})$$
$$= 2^{-l}\delta_{\mathrm{init}}(\boldsymbol{\mu}^k)^T\mathbf{d} + \frac{\epsilon}{l+1}\ell^*(\bar{x}) \leq 2\epsilon\ell^*(\bar{x})$$

for any $\tau = 0, \ldots, N-1$, where $H = \mathrm{blkdiag}(Q, R)$. Taking the square-root and applying the reversed triangle inequality gives

$$\left|\left\|\begin{bmatrix}\xi_\tau^k \\ v_\tau^k\end{bmatrix}\right\|_H - \left\|\begin{bmatrix}z_\tau^* \\ v_\tau^*\end{bmatrix}\right\|_H\right| \leq \left\|\begin{bmatrix}\xi_\tau^k \\ v_\tau^k\end{bmatrix} - \begin{bmatrix}z_\tau^* \\ v_\tau^*\end{bmatrix}\right\|_H \leq 2\sqrt{\epsilon\ell^*(\bar{x})}. \tag{43}$$

This implies that

$$\left\|\begin{bmatrix}\xi_{N-1}^k\\v_{N-1}^k\end{bmatrix}\right\|_H \leq \left\|\begin{bmatrix}z_{N-1}^*\\v_{N-1}^*\end{bmatrix}\right\|_H + 2\sqrt{\epsilon\ell^*(\bar{x})}$$

$$= \sqrt{2}\sqrt{\ell(z_{N-1}^*,v_{N-1}^*)} + 2\sqrt{\epsilon\ell^*(\bar{x})}$$

$$\leq \sqrt{2\Phi_N}\sqrt{\ell(z_0^*,v_0^*)} + 2\sqrt{\epsilon\ell^*(\bar{x})}$$

$$\leq (\sqrt{2\Phi_N} + 2\sqrt{\epsilon})\sqrt{\ell(z_0^*,v_0^*)}$$

$$= (\sqrt{\Phi_N} + \sqrt{2\epsilon})\left\|\begin{bmatrix}z_0^*\\v_0^*\end{bmatrix}\right\|_H$$

$$\leq (\sqrt{\Phi_N} + \sqrt{2\epsilon})\left(\left\|\begin{bmatrix}\xi_0^k\\v_0^k\end{bmatrix}\right\|_H + 2\sqrt{\epsilon\ell^*(\bar{x})}\right)$$

$$\leq (\sqrt{\Phi_N} + \sqrt{2\epsilon})(1 + \sqrt{2\epsilon})\left\|\begin{bmatrix}\xi_0^k\\v_0^k\end{bmatrix}\right\|_H$$

where we have used (43), $z_0^* = \xi_0^k = \bar{x}$, $\|[z^T v^T]^T\|_H = \sqrt{z^T Q z + v^T R v} = \sqrt{2\ell(z,v)}$ and Definition 1. Squaring both sides gives through the definition of κ that

$$\frac{1}{\kappa}\ell^*(A\xi_{N-1}^k) \leq \ell^*(\xi_{N-1}^k) = \ell(\xi_{N-1}^k, v_{N-1}^k)$$

$$\leq (\sqrt{\Phi_N} + \sqrt{2\epsilon})^2(1 + \sqrt{2\epsilon})^2\ell(\xi_0^k, v_0^k). \quad (44)$$

We get for large enough l and for k such that (40) holds that

$$D_N^\delta(\bar{x}, \boldsymbol{\lambda}^k, \boldsymbol{\mu}^k) \geq$$

$$\geq P_N(\bar{x}, \mathbf{v}^k) - \frac{\epsilon}{l+1}\ell^*(\bar{x})$$

$$\geq P_N(\bar{x}, \mathbf{v}^k) - \epsilon\ell^*(\bar{x})$$

$$= P_N(A\bar{x} + Bv_0^k, \mathbf{v}_s^k) + (1-\epsilon)\ell(\xi_0^k, v_0^k) - \ell^*(A\xi_{N-1}^k)$$

$$\geq P_N(A\bar{x} + Bv_0^k, \mathbf{v}_s^k) +$$

$$\quad + \left(1 - \epsilon - \kappa(\sqrt{\Phi_N} + \sqrt{2\epsilon})^2(1 + \sqrt{2\epsilon})^2\right)\ell(\bar{x}, v_0^k)$$

$$\geq P_N(A\bar{x} + Bv_0^k, \mathbf{v}_s^k) + \alpha\ell(\bar{x}, v_0^k)$$

where the first inequality comes from (40), the second since $l \geq 0$, the equality is due to (16), the third inequality comes from (44), and the final inequality comes from (25). This concludes the proof for argument 2. Thus, the do loop will terminate with finite l and k. This implies that ν_N is defined for every $\bar{x} \in \text{int}(\mathbb{X}_N^0)$, i.e. that $\text{dom}(\nu_N) \supseteq \text{int}(\mathbb{X}_N^0)$.

Finally, to show (26) we have that

$$V_N^0(\bar{x}) \geq D_N^\delta(\bar{x}, \boldsymbol{\lambda}^k, \boldsymbol{\mu}^k) - \delta\mathbf{d}^T\boldsymbol{\mu}^k$$

$$\geq P_N(A\bar{x} + Bv_0^k, \mathbf{v}_s^k) - \epsilon\ell^*(\bar{x}) + \alpha\ell(\bar{x}, v_0^k)$$

$$\geq V_N^0(A\bar{x} + Bv_0^k) + (\alpha - \epsilon)\ell(\bar{x}, v_0^k)$$

where the first inequality comes from Lemma 1, the second from (38) and (39) which obviously hold also for any $\bar{x} \in \text{dom}(\nu_N)$, and the third holds since $P_N(A\bar{x} + Bv_0^k, \mathbf{v}_s^k) \geq V_N(A\bar{x} + Bv_0^k)$ and by definition of ℓ^*. This concludes the proof. $\qquad\square$